Motion of test particles in six-dimensional dilatonic Kaluza-Klein theory

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Multidimensional theories still remain attractive from the point of view of better understanding of fundamental interactions. In this paper we consider a six-dimensional Kaluza–Klein type model at the classical level. We derive static spherically symmetric solutions to the multidimensional Einstein equations. They are fundamentally different from four-dimensional Schwarzschild solutions: they are horizon free and the presence of massless dilaton field has the same dynamical effect as the existence of additional massive matter in the system. Then we analyse the motion of test particles in such spherically symmetric configurations. The emphasis is put on some observable quantities like redshifts. It is suggested that strange features of emission lines from active galactic nuclei as well as quasar–galaxy associations may in fact be manifestations of multidimensionality of our world.

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I. INTRODUCTION

And do not be called teachers; for One is your Teacher, the Christ Jesus.

Holy Bible, Matthew 23:10

Many recent ideas in theoretical physics assume the possibility that our world may have more than four dimensions. The discovery of the cancellation of infinities in superstring theories has stimulated the interest in higher-dimensional theories and their compactification. The renewed interest in Kaluza-Klein theories stems from the fact that multidimensional analogues of general relativity are able to generate non-Abelian gauge theories out of theories with symmetries of compact internal space. Multidimensional versions of supersymmetric theories are examples of another approach to construct a theory of elementary interactions [1],[2],[3] by exploiting the formal symmetry between bosons and fermions. These theories could provide better description of observational aspects of the world we see. However, the main problem of multidimensional models is their complexity reflected either in analytical or numerical studies — even in the case of models with the Ricci flat Calabi-Yau manifolds.

Up to now there is no well established experimental evidence of six-dimensionality of our world and our under-
standing of potential manifestations of higher dimensions is too poor. But there were also attempts in the literature to seek the effects of extra dimensions in the astrophysical setting pursued actively by Wesson \[4\], Lim, Kalligas, Everitt, Biesiada \[5, 6\], Mańska, and Syska. This line of thinking is worth of developing in order to gain better understanding concerning possible manifestations of six-dimensionality of the world. In particular this would mean that the effects of extra dimensions may be well around us on the contrary of standard expectations that extremely high energies are necessary to probe the higher dimensions.

Hence the contents of the present paper suggest that there is an intimate relation between the manifestations of six-dimensionality of this world and the so-called dark matter. But the first evidence for the existence of the dark matter in galaxies came into play when Oort in 1932 \[7\] and Zwicky in 1933 have applied the famous virial theorem to vertical motion of stars in the Galaxy and to the radial velocities of members of Coma cluster, respectively. The problem revived and become well established in the seventies when it has been demonstrated \[8, 9, 10\] that the rotation curves of spiral galaxies were indicative to the presence of unseen \(^1\) (in any part of electromagnetic spectrum) mass \(^2\). In their paper \[11\] Neta Bahcall, Lubin and Dorman point to the evidence that most dark matter in the universe resides in large dark halos around galaxies.

Thus the present paper provides a description of properties of certain six-dimensional Kaluza-Klein type model with pointing to some of possible observational consequences. Section II contains a description of the model along with motivations for the choice of six-dimensional spacetime. Then we derive static spherically symmetric solutions of the multidimensional Einstein equations \[5\]. They are in a sense analogous to the familiar four-dimensional Schwarzschild solution but fundamentally different i.e. they are horizon free. More detailed discussion of the properties of these solutions is presented in Sections II and III. We also discuss briefly some strictly observable quantities in the model such like the redshift formulae. Section IV is devoted to the analysis of motion of test particles. Finally Section V contains concluding remarks and perspectives. Some more formal although important remarks are put into Appendices in the end of this paper. Throughout Section II we are using natural units \((c = \hbar = 1)\) whereas in Sections III, IV and V which deal with some more observationally oriented issues we reintroduce explicitly the velocity of light (the Planck constant turns out to be irrelevant for these considerations).

\(^1\) The word unseen would perhaps reflect better the nature of the missing mass being free from potential negative connotations associated with the word dark. However in the rest of this work we shall use the traditional notion of dark matter.

\(^2\) There are currently many candidates for the dark matter with massive compact halo objects (MACHOs) such like "brown dwarfs", "jupiters" etc. on one extreme to hypothetical elementary particles such like massive neutrinos, axions and other weakly interacting massive particles (WIMPs) which till now are either unable to give the amount of needed mass or are in contradiction with other parts of proposed models.
II. FIELD EQUATIONS

Let us consider a six-dimensional field theory comprising the gravitational self field described by a metric tensor, $g_{MN}$, and a real massless "basic" scalar field, $\varphi$. The scalar field $\varphi$ is a dilaton field—hence, just below, the minus sign in front of its kinetic energy term [5, 6]. In a standard manner we decompose the action into two parts:

$$S = S_{EH} + S_\varphi,$$

(II.1)

where $S_{EH}$ is the Einstein — Hilbert action

$$S_{EH} = \int \frac{1}{2\kappa_6} \sqrt{-g} \mathcal{R} \, d^6x$$

(II.2)

and $S_\varphi$ is the action for a real massless scalar field

$$S_\varphi = -\int \sqrt{-g} \left\{ \frac{1}{2} g_{MN} \partial^M \varphi \partial^N \varphi - \frac{1}{4} \kappa \exp(\sqrt{2}\kappa \varphi) F_{PQ} F^{PQ} - \frac{1}{2} \left( \frac{g_0^3}{\kappa^3} \right) \exp(-\sqrt{2}\kappa \varphi) \right\} \, d^6x.$$  

(II.3)

In the above equation, $F_{PQ} = \partial_P A_Q - \partial_Q A_P$, $\varphi$, and $A_P (P = 0, ..., 6)$ are boson fields, $g_0$ is the U(1) coupling constant. The mathematical compactification of the full six-dimensional space into a direct product $M^4 \times S^2$ is realized with the ground state expectation values: $\varphi = \varphi_0$, and $F_{MN} = F_{mn} = -\frac{1}{2} g_0 b^2 \varepsilon_{mn}$ for $M, N = m, n$, and 0 otherwise ($b$ is the radius of the microspace). Our model is similar to the above model (considered in Refs. [12] and [13]), in absence of boson fields$^4$. A six-dimensional model of the Kaluza-Klein theory was also previously investigated by Ivashchuk and Melnikov [14, 15], Bronnikov and Melnikov [15], and by one of us in [17].

The simplest extension of familiar four-dimensional spacetime is to consider five-dimensional models, as Wesson considered in [4]. We have chosen a six-dimensional theory which is much more robust and interesting.

By extremalizing the action given by Eqs.(II.2)-(II.3) their Lagrangian density was equal to

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_m,$$

where:

$$\mathcal{L}_{EH} = \frac{1}{4 \kappa^2} \mathcal{R}$$

$$\mathcal{L}_m = -\frac{1}{2} g^{MN} \partial_M \varphi \partial_N \varphi - \frac{1}{4} \kappa \exp(\sqrt{2}\kappa \varphi) F_{PQ} F^{PQ} - \frac{1}{2} \left( \frac{g_0^3}{\kappa^3} \right) \exp(-\sqrt{2}\kappa \varphi),$$

In the above equation, $F_{PQ} = \partial_P A_Q - \partial_Q A_P$, $\varphi$, and $A_P (P = 0, ..., 6)$ are boson fields, $g_0$ is the U(1) coupling constant. The mathematical compactification of the full six-dimensional space into a direct product $M^4 \times S^2$ is realized with the ground state expectation values: $\varphi = \varphi_0$, and $F_{MN} = F_{mn} = -\frac{1}{2} g_0 b^2 \varepsilon_{mn}$ for $M, N = m, n$, and 0 otherwise ($b$ is the radius of the microspace). Our model is similar to the above model (considered in Refs. [12] and [13]), in absence of boson fields$^4$. A six-dimensional model of the Kaluza-Klein theory was also previously investigated by Ivashchuk and Melnikov [14, 15], Bronnikov and Melnikov [15], and by one of us in [17].

The simplest extension of familiar four-dimensional spacetime is to consider five-dimensional models, as Wesson considered in [4]. We have chosen a six-dimensional theory which is much more robust and interesting.

By extremalizing the action given by Eqs.(II.2)-(II.3) the inclusion of gauge fields into the model and its consequences for the problem of dark matter will be a subject of forthcoming papers.

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$^3$ This scalar field $\varphi$ is the created fluctuation out from nothingness.

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we obtain the Einstein equations

\[ R_{MN} - \frac{1}{2} g_{MN} R = \kappa_6 T_{MN}. \]  

(II.4)

Here \( R_{MN} \) is the six-dimensional Ricci tensor, \( R \) is the six-dimensional curvature scalar and \( T^M_N \) is the energy-momentum tensor of a real scalar field \( \varphi \) which is given by

\[ T^M_N = \partial_N \varphi \frac{\partial L_\varphi}{\partial (\partial_M \varphi)} - \delta^M_N L_\varphi. \]  

(II.5)

Variation of the total action \( S \) with respect to the field \( \varphi \) gives the Klein-Gordon equation

\[ \Box \varphi = 0 , \]  

(II.6)

where

\[ \Box = -\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} \partial_N) \]

and \( g^{MN} \) is the tensor dual to \( g_{MN} \).

Now, we assume that we live in the compactified (which is quite a reasonable assumption) world, where the six-dimensional spacetime is a topological product of “our” curved four-dimensional physical spacetime (with the metric \( g_{\alpha\omega} \), \( \alpha, \omega = 0, 1, 2, 3 \)) and the internal space (with the metric \( g_{he} \), \( h, e = 5, 6 \)).

Therefore the metric tensor can be factorized as

\[ g_{MN} = \begin{pmatrix} g_{\alpha\omega} & 0 \\ 0 & g_{he} \end{pmatrix}. \]  

(II.7)

The four-dimensional diagonal part is assumed to be that of a spherically symmetric geometry

\[ g_{\alpha\omega} = \begin{pmatrix} e^{\nu(r)} & -e^{\mu(r)} & 0 \\ -e^{\mu(r)} & 0 & -r^2 \\ 0 & -r^2 \sin^2 \Theta \end{pmatrix}, \]  

(II.8)

where \( \nu(r) \) and \( \mu(r) \) are (at this stage) two arbitrary functions.

Analogously, we take the two-dimensional internal part to be

\[ g_{he} = \begin{pmatrix} -g^2(r) \cos^2 \vartheta & 0 \\ 0 & -g^2(r) \end{pmatrix}. \]  

(II.9)

The six-dimensional coordinates \( (x^M) \) are denoted by \( (t, r, \Theta, \Phi, \vartheta, \varsigma) \) where \( t \in [0, \infty) \) is the usual time coordinate, \( r \in [0, \infty), \Theta \in [0, \pi] \) and \( \Phi \in [0, 2\pi) \) are familiar three-dimensional spherical coordinates in the macroscopic space; \( \vartheta \in [-\pi, \pi) \) and \( \varsigma \in [0, 2\pi) \) are coordinates in the internal two-dimensional space and \( g \in (0, \infty) \) is the “radius” of this two-dimensional internal space. We assume that \( g(r) \) is the function of the radius \( r \) in our external three-dimensional space\(^5\).

The internal space is a 2-dimensional topological torus with \( r \)-dependent parameter \( g(r) \) which can be represented as a surface embedded in the three-dimensional Euclidean space

\[ \begin{cases} w^1 = g(r) \cos \varsigma, & \varsigma \in [0, 2\pi) \\ w^2 = g(r) \sin \varsigma \\ w^3 = \varrho \cos \vartheta, & \vartheta \in [-\pi, \pi). \end{cases} \]  

(II.10)

Now using Eqs.(II.8)-(II.9), we can calculate the components of the Ricci tensor. The nonvanishing components are

\[ R^r_r = \left( 4 \varrho^2 r \nu' + 4 r^2 \varrho^2 \nu' - r^2 \varrho^2 \mu' \nu' + r^2 \varrho^2 (\nu')^2 + 2 r^2 \varrho^2 (\mu')^2 \right) (4 \nu r^2 \varrho^2)^{-1} \]  

(II.11)

\(^5\) special thanks to Ryszard Mańka for pointing to this direction
\[ R^r_r = (-4 \, g^2 r \mu' - 4 r^2 g' g \mu' - r^2 g^2 \mu' \nu' + 
+ r^2 g^2 (\nu')^2 + 8 r^2 g g'' + 
+ 2 r^2 g^2 \nu') (4 e^{\nu r^2 g^2})^{-1} \]  

\[ R^\phi_{\phi} = R^\vartheta_{\vartheta} = R^\varphi_{\varphi} = R^\zeta_{\zeta} = (-4 e^{\mu} g^2 + 4 g^2 + 8 r g g' - 
- 2 r g^2 \mu' + 2 r g^2 \nu') (4 e^{\nu r^2 g^2})^{-1} \]

\[ R_\varphi^\varphi = R_\vartheta^\vartheta = R_\varphi^\varphi = R_\zeta^\zeta = (8 g r g' + 4 r^2 (g')^2 - 2 r^2 g g' \mu' + 
+ 2 r^2 g g' \nu' + 4 r^2 g g' \nu') (4 e^{\nu r^2 g^2})^{-1} \]

Let us assume that we are looking for a solution of the Einstein equations (see Eq. (II.4)) with the Ricci tensor given by Eqs. (II.11)-(II.14), with \( \nu(r) = \mu(r) \), and with the following boundary conditions

\[ \lim_{r \to \infty} \nu(r) = \lim_{r \to \infty} \mu(r) = 0 \]  

\[ \lim_{r \to \infty} \varphi(r) = d = constant \neq 0 \]  

In other words, we are looking for the solution which at spatial infinity reproduces the flat external four-dimensional Minkowski spacetime and static internal space of “radius” \( d \) which is of order of \( 10^{-33} \, m \).

Now we make an assumption that the scalar field \( \varphi \) depends neither on time \( t \) nor on internal coordinates \( \psi \) and \( \zeta \). Now, because of assumed spherical symmetry of the physical spacetime, it is natural to suppose that the scalar field \( \varphi \) is the function of the radius \( r \) alone, \( \varphi = \varphi(r) \). We impose also the boundary condition for the scalar field \( \varphi \)

\[ \lim_{r \to \infty} \varphi(r) = 0 \]

which supplements boundary conditions (II.15) and (II.16) for the metric components.

By virtue of Eqs. (II.5) and (II.3) it is easy to see that the only nonvanishing components of the energy-momentum tensor are

\[ - T^r_r = T^i_i = T^\phi_\phi = T^\vartheta_\vartheta = T^\varphi_\varphi = T^\zeta_\zeta = 
\frac{1}{2} g^{rr} (\partial_r \varphi)^2. \]

Consequently, it is easy to verify that the solution of the Einstein equations (II.4) is

\[ \nu(r) = \mu(r) = \ln \left( \frac{r}{r + A} \right) \]

\[ \varphi(r) = d \sqrt{\frac{r + A}{r}} \]

\[ \varphi(r) = \pm \sqrt{\frac{1}{2} \kappa_6} \ln \left( \frac{r}{r + A} \right). \]

Hence we obtain that the only nonzero component of the Ricci tensor (see Eqs. (II.11)-(II.14)) is \( R^r_r \) which reads

\[ R^r_r = \frac{A^2}{2 r^3 (r + A)}. \]

So the curvature scalar \( R \) is equal to

\[ R = R^r_r = \frac{A^2}{2 r^3 (r + A)}. \]

where \( A \) is the real constant, with the dimensionality of length, which value is to be taken from observation for each particular system.

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\(^{6}\) Putting Eqs. (II.24), (II.18) and (II.23) together we can notice a similarity between the equation

\[ R^r_r = -\kappa_6 (\partial_r \varphi)^2 g^{rr} \]  

and its electromagnetic analog

\[ \nabla^2 A = m^3_A A, \]

where \( A \) is the electromagnetic vector potential.

Eq. (\( * \)) is the screening current condition in gravitation, analogous to that in electromagnetism.
In derivation of the above solutions, we have used Eqs. (II.11)-(II.14) which together with Eq. (II.18), imply that all of the six diagonal Einstein equations are equal to just one
\[ \frac{1}{2} R = \kappa_6 T^r_r. \]  
(II.24)

Now we can rewrite the metric tensor in the form
\[
g_{MN} = \text{diag} \left( \frac{r}{r + A}, -\frac{r}{r + A}, -r^2, -r^2 \sin^2 \Theta, -d^2 \frac{r + A}{r}, -d^2 \frac{r + A}{r} \right) 
\] 
(II.25)
with its determinant equal to
\[
g = \det g_{MN} = -(d^2 r^2 \sin \Theta \cos \vartheta)^2. \] 
(II.26)

Thus, we see that the spacetime of our model is stationary.

It is also necessary to verify, whether the solution of the Klein-Gordon equation (see Eq. (II.6)) is in agreement with Eq. (II.21), which follows from the Einstein equations. From Eq. (II.6), we obtain that
\[
\partial_r \varphi(r) = -C g_{rr} r^{-2} = C \frac{1}{r(r + A)}, \]  
(II.27)
where C is a constant. Comparing this result with Eq. (II.21) we conclude that if
\[
C = \pm \frac{A}{\sqrt{2 \kappa_6}} \] 
(II.28)
then the solution of the Klein-Gordon equation is in agreement with the solution of the Einstein equations coupled to Klein-Gordon equation. Hence, the real massless "basic" free scalar field \( \varphi(r) \) (see Eq. (II.21)) can be the source of the nonzero metric tensor as in Eq. (II.25).

Only when the constant \( A \) is equal to zero, the solutions (II.19) – (II.21) become trivial and the six-dimensional spacetime is Ricci flat.

It is worth noting that because the components \( R^0_0 \) and \( R^1_1 \) of the Ricci tensor are equal to zero for all values of \( A \), the internal space is always Ricci flat. However, we must not neglect the internal space because its "radius" \( \varrho \) is a function of \( r \) and the two spaces, external and internal, are therefore "coupled". Only when \( A = 0 \) are these two spaces "decoupled", and our four-dimensional spacetime becomes Minkowski flat.

### III. SOME PROPERTIES OF SOLUTIONS

If the parameter \( A \) is strictly positive, \( A > 0 \), then Eqs. (II.19)-(II.21) are valid for all \( r > 0 \). The metric tensor becomes singular only at \( r = 0 \), nevertheless its determinant \( g \) (see Eq. (II.26)) remains well defined. Below, we shall collect several formulae which will be useful in our later discussion. We start with time and radial components of the metric \( g_{MN} \) and the internal "radius" \( \varrho(r) \) (see Eqs. (II.20) and (II.25))
\[
\begin{align*}
g_{tt} &= \frac{r}{r + A} \\
g_{rr} &= -\frac{r}{r + A} \\
g(r) &= d\sqrt{\frac{r + A}{r}}.
\end{align*}
\]  
(III.1)

\[ ^7 \text{When } A \text{ is not equal to zero our four-dimensional external spacetime is curved and its scalar curvature } R_4 \text{ could be given by Eq. (II.23)} \]
\[ R_4 = R = \frac{A^2}{2 r^3 (r + A)}. \]  
(II.29)

\[ ^8 \text{The discussion of } A < 0 \text{ case will be left for the Appendix A} \]
Table I: Values of the $A$ parameter for which the six-dimensional world influences the dynamics of test particles in a similar way as the existence (in the 4-dimensional world) of mass $M$ given for some examples motivated by astrophysics. The third example and the last example are exceptional in the sense that the $A$ parameter has been estimated by demand to explain the observed redshift peculiarities of such systems. Hence the mass $M$ has purely effective meaning here — for details see Sections IV and V.

It will also be useful to write the explicit relation for the real, physical radial distance $r_l$ from the center

$$r_l = \int_0^r dr \sqrt{1 - g_{rr}} = \sqrt{\frac{r}{r + A}} \left(r + A\right) + \frac{1}{2} A \ln \left(\frac{A}{A + 2r + 2(r + A)\sqrt{\frac{r}{r + A}}}\right) < r. \quad (III.2)$$

Because $g_{tt} \rightarrow 1$ for $r \rightarrow \infty$ (see Eq.(III.1)), it is interesting to compare the gravitational potential $g_{tt} = \frac{r}{r + A} \approx 1 - \frac{A}{r}$ for $r \gg A$ with the gravitational potential

$$g_{tt} = 1 - \frac{G}{c^2} \frac{2M}{r}$$

induced by a mass $M$ in the Newtonian limit. $G$ and $c$ are the four-dimensional gravitational constant and the velocity of light, respectively. Comparing these two potentials, we obtain that $A = 2G/M$, so the parameter $A$ can have the same dynamical consequences as the mass $M$. Table I contains some astrophysically interesting masses that mimic the values of the $A$ parameter.

In this case the gravitational potential $g_{tt}$ (see Eq.(II.25) or Eq.(III.1)) is attractive although there is no massive matter acting as a source.

Let us recall that, in standard derivation of the Schwarzschild solution, the free parameter in the metric tensor is identified with the total mass of spherically symmetric configuration by the demand that at large distances the metric tensor should reproduce the Newtonian potential. In our case, we cannot identify the $A$ parameter directly with $M$. The reason is that our solution describes the case where ordinary matter is absent. The only contribution to the energy-momentum tensor comes from the dilaton (massless) scalar field $\varphi$.

It is well known (see [18]) that the frequency $\omega_0$ of light, moving along the geodesic line in gravitational field which is static or stationary, measured in the units of time $t$, is constant ($\omega_0 = \text{constant}$) along the geodesic. The frequency $\omega$ of light as a function of the proper time $\tau$ ($d\tau = \sqrt{g_{tt}} \, dt$) is equal to

$$\omega = \omega_0 \, \frac{dt}{d\tau} = \omega_0 \sqrt{g_{tt}} = \omega_0 \sqrt{g^{tt}}. \quad (III.3)$$

Let us assume that photon with frequency $\omega_s$ (measured in units of the proper time $\tau$) is emitted from the source which is located at a point $r = r_s$ where $g_{tt} = g_{tt}^s$. Then the photon is moving along a geodesic line and it reaches the observer at the point $r = r_{obs}$, where $g_{tt} = g_{tt}^{obs}$, with the frequency $\omega_{obs}$

$$\frac{\omega_{obs}}{\omega_s} = \sqrt{\frac{g_{tt}^s}{g_{tt}^{obs}}} = \sqrt{\frac{g_{tt}^s}{g_{tt}^{obs}}}. \quad (III.4)$$

Using Eq.(III.1) we can rewrite Eq.(III.4) as

$$\frac{\omega_{obs}}{\omega_s} = \sqrt{\frac{r_s}{r_s + A}} \frac{r_{obs}}{r_{obs} + A}. \quad (III.5)$$
For simplicity, let us consider the limiting case when the observer is situated at the infinity. Then we get
\[
\frac{\omega_{\text{obs}}}{\omega_s} = \sqrt{\frac{r_s^w}{r_s^w + 1}}, \text{ where } r_s^w = \frac{r_s}{A}. \tag{III.6}
\]
Therefore, we obtain that the nearer is the source to the center of the field \(\varphi(r)\) the more is the emitted photon redshifted at the point where it reaches the observer (It should also be emphasized that this redshift does depend on the relative radius \(r_w = \frac{r}{A}\) rather than separately on \(r\) and \(A\) (see also Appendix B)).

IV. SIX-DIMENSIONAL HAMILTON-JACOBI EQUATION FOR A “TEST PARTICLE”

In order to gain better understanding concerning possible manifestations of six-dimensionality of the world let us investigate the motion of a “particle” with a mass \(m \neq 0\) in the central gravitational field described by the Eq.(II.25). Whether the moving object can be treated as a “test particle” depends on the value of the \(A\) parameter in the metric tensor \(g_{MN}\) (see Eq.(II.25)). Here \(m\) is the mass of a “test particle” in the six-dimensional spacetime.

The Hamilton-Jacobi equation
\[
g^{MN} \partial_M S \partial_N S - m^2 c^2 = 0 \tag{IV.1}
\]
describing the motion of a test particle reads
\[
\frac{r + A}{r} \left( \frac{\partial S}{\partial t} \right)^2 - \frac{r + A}{r} \left( \frac{\partial S}{\partial r} \right)^2 - \frac{1}{r^2} \left( \frac{\partial S}{\partial \Theta} \right)^2 - \frac{1}{r^2 \sin^2 \Theta} \left( \frac{\partial S}{\partial \Phi} \right)^2 - \frac{1}{d^2} \frac{r}{r + A} \frac{1}{\cos^2 \theta} \left( \frac{\partial S}{\partial \theta} \right)^2 - \frac{1}{d^2} \frac{r}{r + A} \left( \frac{\partial S}{\partial \kappa} \right)^2 - m^2 c^2 = 0, \tag{IV.2}
\]
where \(S\) denotes the action\(^\text{10}\).

Without lack of generality we shall restrict ourselves to the motion in the plane \(\Theta = \pi/2\), thus
\[
\frac{\partial S}{\partial \Theta} = 0. \tag{IV.3}
\]

The standard procedure of the separation of variables begins with the following factorization of \(S\)
\[
S = -E_0 t + M_\Phi \Phi + S_r(r) + M_\kappa \kappa + S_\theta(\theta). \tag{IV.4}
\]

Then by separating Eq.(IV.2) into four-dimensional and internal parts one arrives at the formula
\[
\frac{r + A}{r} \left[ \frac{r + A}{r} \left( \frac{E_0}{c} \right)^2 - \frac{r + A}{r} \left( \frac{\partial S}{\partial r} \right)^2 - \frac{1}{r^2} \frac{1}{\sin^2 \Theta} \left( \frac{\partial S}{\partial \Phi} \right)^2 - \frac{1}{d^2} \frac{1}{\cos^2 \theta} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{d^2} \frac{1}{M_\kappa^2} \right] = 1 \tag{IV.5}
\]
\[
\frac{1}{d^2} \frac{1}{\cos^2 \theta} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{d^2} \frac{1}{M_\kappa^2} =: k_\theta^2 = \text{constant}. \tag{IV.6}
\]

The last equation in (IV.5) is easy to integrate for \(S_\theta\)
\[
S_\theta = \pm \left( d^2 k^{2}_\kappa - M_\kappa^2 \right)^{1/2} \sin \theta = \pm k_\theta \sin \theta. \tag{IV.7}
\]

The separation constants \(E_0, M_\Phi, M_\kappa\) and \(k_\theta\) have the meaning of the total energy, the effectual angular momentum, the internal angular momentum and internal momentum respectively. One may also distinguish the internal total momentum \(k_\kappa\)
\[
k^{2}_\kappa = \frac{M_\kappa^2}{d^2} + k_\theta^2. \tag{IV.8}
\]

Analogously the quantity
\[
m^2_\kappa^2 = m^2 + \frac{k^{2}_\kappa}{c^2}. \tag{IV.9}
\]
\(^{10}\) The action \(S\) has the meaning of the complete integral of the Hamilton-Jacobi equation and should not be confused with six-dimensional field-theoretical action \(S\) invoked in Section II.
can be interpreted as the four-dimensional squared mass of a test particle in the flat Minkowskian limit at infinity. In the case of vanishing internal momentum \( k_{\varnothing c} = 0 \) the four and six-dimensional masses are equal \( m_4 = m \). If the six-dimensional mass is zero \( m = 0 \) then the four-dimensional mass at spatial infinity would be solely of internal origin: \( m_4 = \frac{|k_{\varnothing c}|}{c} \).

Let us now consider the radial part \( S_r(r) \) which can be easily read off from the Eq. (IV.5)

\[
\frac{dS_r}{dr} = \left[ \frac{\mathcal{E}_0^2}{c^2} - \left( m^2 c^2 + \frac{\mathcal{M}_\Phi^2}{r^2} \right) \right] \frac{r}{r + A} - k_{\varnothing c}^2 \left( \frac{r}{r + A} \right)^{\frac{1}{2}},
\]

or after formal integration

\[
S_r(r) = \int dr \left[ \frac{\mathcal{E}_0^2}{c^2} - \left( m^2 c^2 + \frac{\mathcal{M}_\Phi^2}{r^2} \right) \right] \frac{r}{r + A} - k_{\varnothing c}^2 \left( \frac{r}{r + A} \right)^{\frac{1}{2}}. 
\]

The trajectory of a test particle is implicitly determined by the following equations

\[
\frac{\partial S}{\partial \mathcal{E}_0} = \alpha_1 = -t + \frac{\partial S_r(r)}{\partial \mathcal{E}_0} \quad \text{ (IV.11)}
\]

\[
\frac{\partial S}{\partial \mathcal{M}_\Phi} = \alpha_2 = \Phi + \frac{\partial S_r(r)}{\partial \mathcal{M}_\Phi} \quad \text{ (IV.12)}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are constants and without lack of generality the initial conditions can be chosen so that \( \alpha_1 = \alpha_2 = 0 \). In other words integration of Eq. (IV.12) gives the trajectory \( r = r(\Phi) \) of a test particle and Eq. (IV.11) provides the temporal dependence of radial coordinate \( r = r(t) \). These two relations determine the trajectory of a test particle \( r = r(t) \) and \( \Phi = \Phi(t) \) [19].

Implementing the above outlined procedure we obtain (from Eq. (IV.11) and Eq. (IV.10))

\[
t = \frac{\mathcal{E}_0}{c^2} \int dr \left[ \frac{\mathcal{E}_0^2}{c^2} - \left( m^2 c^2 + \frac{\mathcal{M}_\Phi^2}{r^2} \right) \right] \frac{r}{r + A} - k_{\varnothing c}^2 \left( \frac{r}{r + A} \right)^{\frac{1}{2}} \quad \text{ (IV.13)}
\]

and consequently the radial velocity

\[
\frac{dr}{dt} = \frac{c^2}{\mathcal{E}_0} \left[ \frac{\mathcal{E}_0^2}{c^2} - \left( m^2 c^2 + \frac{\mathcal{M}_\Phi^2}{r^2} \right) \right] \frac{r}{r + A} - k_{\varnothing c}^2 \left( \frac{r}{r + A} \right)^{\frac{1}{2}}. \quad \text{ (IV.14)}
\]

It is easy to notice that the quantity

\[
m_4(r) = \sqrt{m^2 \left( \frac{r}{r + A} \right) + k_{\varnothing c}^2 \left( \frac{r}{r + A} \right)^2} \quad \text{ (IV.15)}
\]

can be interpreted as the mass in the four-dimensional curved spacetime. The \( m_4 \) of Eq. (IV.8) is recovered as the limit of \( m_4(r) \) for \( r \to \infty \). Similarly to our previous discussion, if \( k_{\varnothing c} = 0 \) then \( m_4(r) = m \sqrt{\frac{r}{r + A}} \) and if \( m = 0 \) then the mass \( m_4(r) \) would have the internal origin \( m_4(r) = \frac{|k_{\varnothing c}|}{c} \left( \frac{r}{r + A} \right) \). It should also be emphasized that \( m_4(r) \) does depend on the ratio \( \frac{r}{A} \) rather than separately on \( r \) and \( A \), reflecting the scale-invariance of the model (see also Appendix B).

In a similar manner using the Eq. (IV.12) and Eq. (IV.10) we obtain

\[
\Phi = \int \left[ \frac{\mathcal{E}_0^2}{c^2} - \left( m^2 c^2 + \frac{\mathcal{M}_\Phi^2}{r^2} \right) \right] \frac{r}{r + A} - (k_{\varnothing c}^2) \left( \frac{r}{r + A} \right)^{\frac{1}{2}} \frac{\mathcal{M}_\Phi dr}{r(r + A)} \quad \text{ (IV.16)}
\]

and consequently using Eqs. (IV.16) and (IV.14) the angular velocity of the particle reads

\[
\Omega_t = \frac{d\Phi}{dt} = \frac{d\Phi}{dr} \frac{dr}{dt} = \frac{c^2}{\mathcal{E}_0 r(A + r)} \quad \text{ (IV.17)}
\]
Now the proper angular velocity of the particle is equal to
\[ \Omega_{\tau} = \frac{d\Phi}{d\tau} = \Omega_{t} \sqrt{\frac{r + A}{r}}, \] (IV.18)
where \( \tau \) is the proper time and \( d\tau = \sqrt{g_{tt}} \, dt \) (see Eq. (II.25)).

The transversal velocity of the particle (i.e. the component perpendicular to the radial direction) is equal to (see Eq. (III.2))
\[ v_{t} = \Omega_{t} \, r_{t} \] (IV.19)
and analogously the transversal component of the proper velocity (written in the units of proper time \( \tau \)) is equal to
\[ v_{\tau} = \Omega_{\tau} \, r_{\tau} = v_{t} \sqrt{\frac{r + A}{r}} \] (IV.20)
where \( \Omega_{t} \), \( \Omega_{\tau} \) and \( r_{t} \) are given by Eqs. (IV.17), (IV.18) and (III.2) respectively.

Let us rewrite Eq. (IV.14) in the following form
\[ \frac{dr}{dt} = \frac{c}{\mathcal{E}_{0}} \sqrt{\mathcal{E}_{0}^{2} - \mathcal{U}_{eff}(r)} \] (IV.21)
where
\[ \mathcal{U}_{eff}(r) = \left[ (m^{2}c^{4} + \left( \frac{\mathcal{M}_{\Phi}}{r} \right)^{2} c^{2}) \frac{r}{r + A} + \right. \]
\[ \left. + \left( k^{2}_{\varphi}c^{2} \right) \left( \frac{r}{r + A} \right)^{2} \right]^{\frac{1}{2}}. \] (IV.22)
The function \( \mathcal{U}_{eff}(r) \) plays the role of the effective potential energy in the sense that the relation between \( \mathcal{E}_{0} \) and \( \mathcal{U}_{eff}(r) \) determines the allowed regions of the motion of the particle.

The proper radial velocity of the particle is equal to
\[ v_{r} = \frac{dr_{t}}{d\tau} = \frac{\sqrt{-g_{rr}} \, dr}{\sqrt{g_{tt}} \, dt} = \frac{dr}{dt}, \] (IV.23)
where in the last equality we used the relation \( g_{tt} = -g_{rr} \) (see Eq. (III.1)) and \( \tau \) is the proper time. So the radial velocity \( dr/dt \) and the proper radial velocity \( v_{r} = dr_{t}/d\tau \) are equal and one should stress that this property is fundamentally different from the analogical relation for the black hole.

A Stable circular orbits.

Let us now consider the motion of a particle with given values of \( \mathcal{M}_{\Phi} \neq 0 \) and \( k_{\varphi} \). Figure 1a illustrates the function \( \mathcal{U}_{eff}(r) \) for different values of \( \mathcal{M}_{\Phi} \) and \( k_{\varphi} = 0 \).

Similarly Figure 1b shows the function \( \mathcal{U}_{eff}(r) \) for different values of \( k_{\varphi} \) with fixed value of \( \mathcal{M}_{\Phi} \).

Let us consider a stable circular orbit with given values of \( \mathcal{E}_{0} \), \( \mathcal{M}_{\Phi} \) and \( k_{\varphi} \). The radius of this orbit can be calculated from the following equations (see Eqs. (IV.21), (IV.22))
\[ \frac{dr}{dt} = 0, \] (IV.24)
\[ \frac{d^{2}r}{dt^{2}} = \frac{dr}{dt} \frac{d}{dt} \frac{dr}{dt} = 0. \] (IV.25)
From Eqs. (IV.25), (IV.21), (IV.22) we obtain the implicit relation between the radius \( r \) of the stable circular orbit, the angular momentum of the particle \( \mathcal{M}_{\Phi} \) and the internal “total momentum” \( k_{\varphi} \)
\[ (\mathcal{M}_{\Phi})^{2} = \left( m^{2}c^{2} + 2 k^{2}_{\varphi} \left( \frac{r}{r + A} \right) \left( \frac{A}{2r + A} \right) \right) r^{2} \] (IV.26)
so
\[ \mathcal{M}_{\Phi} = \pm A^{1/2} r \sqrt{\frac{(r + A) m^{2}c^{2} + 2 k^{2}_{\varphi} r}{A^{2} + 3 A r + 2 r^{2}}} \] (IV.27)
It is easy to verify that \( \mathcal{M}_\Phi/r \to 0 \) as \( r \) tends to infinity.

Using Eqs.(IV.24),(IV.21), (IV.22) we calculate the total energy \( E_0 \) of the particle moving along this stable circular orbit of radius \( r \)

\[
E_0^2 = \left[ m^2 c^4 + \left( m^2 c^4 + \frac{2 \kappa c^2 (\frac{r}{r + A})}{2} \frac{A}{r + A} \right) \left( \frac{r}{r + A} \right) + \frac{k^2 c^2 (\frac{r}{r + A})^2}{(r + A)^2} \right] - \frac{2 A A^2}{r^2 A^2} + \frac{2 A^2}{r^2 A^2}
\]  

or in other words

\[
E_0 = c^2 \sqrt{(r + A)^2 (2 A + r)} \sqrt{2 (r + A)^2 (2 A + r) - \left( \frac{r}{r + A} \right)^2 A^2}. \quad (IV.29)
\]

There are some physically obvious conditions which in turn restrict admissible values of radii of stable orbits for given internal momentum \( k_{\Phi} \), namely

\[
(M_\Phi)^2 > 0 \quad \text{so} \quad k_{\Phi}^2 > -\frac{m^2 c^2}{2} \frac{(r + A)}{r} \quad (IV.30)
\]

\[
E_0^2 \geq 0 \quad \text{so} \quad k_{\Phi}^2 \geq -m^2 c^2 \frac{2 (A + A)}{3 A + 2 r} \frac{r + A}{r + A} \quad (IV.31)
\]

It is not difficult to verify that condition (IV.30) is stronger than (IV.31).

Now let us calculate the proper angular velocity, and the proper transversal velocity of the particle moving along the stable circular orbit given by Eqs.(IV.24),(IV.25). Using Eqs.(IV.17)-(IV.20) with \( \mathcal{M}_\Phi \) and \( E_0 \) given by Eqs.(IV.27), (IV.29) respectively, we obtain the proper angular velocity

\[
\Omega^\tau = \frac{\kappa c^2 \mathcal{M}_\Phi}{E_0 r (A + r)} \sqrt{\frac{r + A}{r}} \quad (IV.32)
\]

and the proper tangent velocity

\[
v^\tau = \frac{\kappa c^2 \mathcal{M}_\Phi}{E_0 r (A + r)} r \sqrt{\frac{r + A}{r}} \quad (IV.33)
\]

Figure 1: (a) The potential \( U_{eff} \) (in units of \( mc^2 \)) (see Eq.(IV.22)) for \( k_{\phi} = 0 \) and different values of \( \mathcal{M}_\Phi \) as a function of the relative radius \( r^w = r/A \). Horizontal arrow above the curves denotes the direction of increasing angular momentum \( \mathcal{M}_\Phi \). The minimum of the potential \( U_{eff} \) determines the radius \( r^w \) (and hence the value of \( r = r^w A \)) of a stable circular orbit. The radii \( r^w \) of stable orbits corresponds to minima of the potentials depicted on Figure 1a are \( 7.5 \times 10^6 \), \( 1.5 \times 10^7 \) and \( 3.1 \times 10^7 \), respectively. The middle curve (continuous) with \( A \) parameter equal to \( 5.10^{-7} \) pc illustrates the effective potential for a stable orbit located at a distance equal to the distance of the Sun from the center of the \( \varphi \) scalar field in our Galaxy i.e. \( r_0 = 7.5 \) kpc.

(b) The potential \( U_{eff} \) (in units of \( mc^2 \)) (see Eq.(IV.22)) for different values of \( k_{\phi} \) and fixed value of \( \mathcal{M}_\Phi \) (chosen for \( k_{\phi} = k_{\phi,min} (r^w = 1/2) \), see Eq.(IV.35) as a function of the relative radius \( r^w = r/A \). The curve for \( k_{\phi,min} (r^w = 1/2) \) (continuous line) has a minimum at \( r^w = 1/2 \) and a maximum at finite \( r^w \) (equal to \( \approx 2.8 \)).
on the stable (index st) circular orbit.

One can notice from Eq. (IV.33) and Eqs. (IV.27), (IV.29) that if \( k_{\varpi}^2 \propto \text{Constant} \cdot m^2 \) then \( v_{\varpi}^\text{st} \) does not depend explicitly on mass \( m \) of the particle. However \( v_{\varpi}^\text{st} \) is still parameterized by the value of proportionality constant which is clearly non-classical effect. It is also not difficult to see that \( v_{\varpi}^\text{st} \) again depends on the ratio \( r_w = \frac{\varpi}{\tau} \) rather then on \( r \) and \( A \) independently.

1 The case with \( m \neq 0 \).

From Eq. (IV.30) (see Figure 2) we may notice that if 
\[ k_{\varpi}^2 > -\frac{1}{2} m^2 c^2 \] 
then all values of the radius \( r \) (or \( r_1 \) cf. 
Eq. (III.2)) for stable orbits are allowed.

If however 
\[ k_{\varpi,\text{min}}^2 \leq k_{\varpi}^2 < -\frac{1}{2} m^2 c^2 \] 
(the lower limit \( k_{\varpi,\text{min}}^2 \) is till now undefined) then according to 
Eq. (IV.30) the stable orbits may exist only up to a radius 
\[ r_0 \equiv -A/(1 + \frac{2k_{\varpi}^2}{m^2 c^2}). \]  

![Figure 2: The proper transversal velocity \( v_{\varpi}^\text{st} \) (see Eq. (IV.33)) of a particle (in units of the velocity of light \( c \)) moving along a stable circular orbit given by Eqs. (IV.24), (IV.25) for different (but fixed for each curve) values of \( k_{\varpi} \) as a function of the relative radius \( r_w = r/A \). If \( k_{\varpi}^2 > -\frac{1}{2} m^2 c^2 \) then all values of the radius \( r_w \) (or \( r_1/A \) cf. Eq. (III.2)) are allowed for stable orbits. If \( k_{\varpi,\text{min}}^2 \leq k_{\varpi}^2 < -\frac{1}{2} m^2 c^2 \) then the stable orbits, for fixed \( k_{\varpi} \) and angular momentum \( M_\varphi \), chosen according to Eq. (IV.27), may exist only up to a radius \( r_{\text{max}} \) (see Eqs. (IV.36), (IV.34)). (As in Figure 1b \( r_{w,\text{max}} = 1/2 \) (then \( r_0^\text{m} = 3.5 \)). For \( m \neq 0 \) the curve drawn in continuous line has physical meaning only up to \( r_{\text{max}} = 1/2 \).

Figure 2 displays the rotation curves calculated according to Eq. (IV.33) (in units of the velocity of light \( c \)) for different values of \( k_{\varpi} \). It shows that the whole effect, mostly pronounced in regions close to the center of spherically symmetric configurations of the field \( \varphi \), is extended from the center to the infinity. (The value of contribution

In other words Eq. (IV.30) implies that one cannot find any stable orbit with \( r \geq r_0 \) for fixed \( k_{\varpi} \) and the angular momentum \( M_\varphi \) chosen according to Eq. (IV.27). From the demand that the value of the effective potential at infinity can be lowered only up to the value of the local minimum of the effective potential at radius \( r \), it follows that the lower limit \( k_{\varpi,\text{min}}^2 \) (which appears to depend on the radius \( r \) \( (r > 0) \)) is equal to
\[ k_{\varpi,\text{min}}^2(r) \equiv -m^2 c^2 \frac{(1 + \frac{1}{2})^2}{1 + \frac{1}{4} + \frac{2}{(2)^2}} > -m^2 c^2. \]  

Now from Eq. (IV.35) we obtain another limit for allowed stable orbits (in the case of \( k_{\varpi,\text{min}}^2 \leq k_{\varpi}^2 < -\frac{1}{2} m^2 c^2 \))
\[ 0 < r_{\text{max}} \equiv r_0 - A \frac{2k_{\varpi}^2 + \sqrt{k_{\varpi}^4(2k_{\varpi}^2 + m^2 c^2)}}{2k_{\varpi}^2 + m^2 c^2} < r_0. \]  

As in Figure 1b \( r_{w,\text{max}} = 1/2 \) (then \( r_0^\text{m} = 3.5 \)). For \( m \neq 0 \) the curve drawn in continuous line has physical meaning only up to \( r_{\text{max}} = 1/2 \).
to total rotational velocity coming from the scalar field \( \varphi \) for the Sun at a distance of 7.5 kpc from the center of the scalar field in our Galaxy for \( k_{\Phi c} = 0 \) is equal to 1.73 km/s.)

2 The case with \( m = 0 \) and \( k_{\Phi c} \neq 0 \).

In the case \( m = 0 \) and \( k_{\Phi c} \neq 0 \) we can see from the Eqs. (IV.30) and (IV.31) that \( k_{\Phi c}^2 > 0 \) and according to our previous discussion all values of \( r \) are allowed.

B Radial trajectories.

In this case we shall investigate free motion of a test particle (with given internal “total momentum” \( k_{\Phi c} \)) along the geodesic \( \Phi = \text{constant} \) (\( M_{\Phi} = 0 \)) crossing the center of the gravitational field \( g_{MN} \). From the Eqs. (IV.21), (IV.22) and (IV.23) we obtain

\[
\ddot{v}_r = \frac{dr}{dt} = \frac{c}{\xi_0} \sqrt{\xi_0^2 - (m^2 c^4) \frac{r}{r + A} - k_{\Phi c}^2 c^2 \left( \frac{r}{r + A} \right)^2}.
\]

From this equation we may see that for the particle which is initially \( (t = t_o) \) at rest \( (v_r = v_{r_o} = 0) \) at the point \( r = r_o \neq 0 \) the total energy is equal to (compare Eq. (IV.15))

\[
\xi_0 = \sqrt{m^2 c^4 \left( \frac{r_o}{r_o + A} \right) + k_{\Phi c}^2 c^2 \left( \frac{r_o}{r_o + A} \right)^2} = m_4(r_o) c^2.
\]

Hence the particle is oscillating and crosses the center with velocity \( v_r(r = 0) = c \) (at the center the particle becomes massless \( m_4(r = 0) = 0 \)). The acceleration of the particle is equal to

\[
a_r = \frac{dv_r}{dr} = \frac{m^2 c^4 + 2 k_{\Phi c}^2 c^2 \left( \frac{r}{r + A} \right)}{2 \xi_0^2 (r + A)^2} \sqrt{\frac{r + A}{r}}.
\]

Because the potential energy at infinity cannot be lower than at the center at \( r = 0 \), then simple comparison of these limits based on formula (IV.22) reveals that \( k_{\Phi c}^2 \geq -m^2 c^2 \), hence \( m_4(r) \geq 0 \) everywhere\(^{11} \) (cf. Eq. (IV.15)).

Note: A remark is in order at this point. Namely it may appear obvious that \( k_{\Phi c}^2 \geq -m^2 c^2 \) (so we admit imaginary values of \( k_{\Phi c} \)). However it is our conviction that one should not uncritically transfer four-dimensional intuitions (such like \( k_{\Phi c}^2 \geq 0 \) or \( m^2 \geq 0 \), although some of them may turn out to be true) but rather build on safe grounds of known properties of four-dimensional sector i.e. \( m_4^2 \geq 0 \) in this case.

1 The case with \( m \neq 0 \).

From Eq. (IV.39) we see that the acceleration tends to minus infinity at the center and when \( k_{\Phi c}^2 > -\frac{1}{2} m^2 c^2 \) it monotonously increases to the zero value when \( r \) is going to infinity. So, in this case, the particle is attracted to the center for all values of \( r \) (see Figure 3a).

In the case of \( -m^2 c^2 \leq k_{\Phi c}^2 < -\frac{1}{2} m^2 c^2 \) there exists the finite value of \( r = r_{a_r=0} \) for which \( a_r = 0 \). For \( r \leq r_o < r_{a_r=0} \) the particle is attracted to the center with \( a_r \to -\infty \) for \( r \to 0 \) and for \( r \geq r_o > r_{a_r=0} \) the

---

\(^{11}\) Because in this case the effective potential is globally the lower one, it follows from this condition that the proportionality constant in equation \( k_{\Phi c}^2 \propto \text{constant} \cdot m^2 \) should be equal to \(-c^2\). Hence we obtain \( k_{\Phi c}^2 = c^2 \cdot m^2 \) as the lower value for all test particles with \( m^2 \neq 0 \), which we [20] believe to be the fundamental law of nature - which we call - the law of the lower potential.
For other curves the particle is attracted to the center for all values of $r$.

---

2 The case with $m = 0$ and $k_{\phi c} \neq 0$.

In this case the requirement of $\epsilon_0^2 > 0$ implies $k_{\phi c}^2 > 0$ (see Eq. (IV.38)). From Eqs. (IV.37), (IV.38) and (IV.39) we conclude that the particle is attracted to the center ($a_r < 0$) for all values of $r$.

---

C The case with $m = 0$ and $k_{\phi c} = 0$.

Now we have $\epsilon_0 = constant \neq 0$ and one can read from Eqs. (IV.37) and (IV.39) that $v_r = c$ and $a_r = 0$ for all values of $r$. So the particle which has both the six-dimensional mass $m$ and the internal “total momentum” $k_{\phi c}$ equal to zero (which is a reasonable representation of a photon for example) does not feel (except of changing the frequency, see Eq. (III.3)) the curvature of the spacetime when moving along the geodesic line crossing the center. On the other hand for $\mathcal{M}_\Phi \neq 0$, $m = 0$ and $k_{\phi c} = 0$ by using the Eq. (IV.16) and introducing formally the parameter $^{12} r_m = \frac{\mathcal{M}_\Phi c}{\epsilon_0}$ we obtain the trajectory of

---

12 We should use for $m = 0$ and $k_{\phi c} = 0$ the eikonal equation instead of the Hamilton-Jacobi equation. However formal (technical) substitution of $r_m = \frac{\mathcal{M}_\Phi c}{\epsilon_0}$ gives the same analytical result.

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Figure 3: (a) The acceleration $a_r$ (in units of $1/A$) of a particle moving along a radial trajectory (see Eq. (IV.39)). As in Figure 1b $k_{\phi c}$ is chosen equal to $k_{\phi c, \min}(r_{\max} = 1/2)$. Consequently $a_r = 0$ for the relative radius $r^w = r^w_0 = 3.5$ (see Eq. (IV.34), (IV.36)) and the “particle” is attracted to the center for all values of $r^w < r^w_0$ and repelled for $r^w > r^w_0$. For simplicity we have chosen $r^w_0 = r_0/A = r^w_0$ (see text, Section IV.B). For other curves the particle is attracted to the center ($a_r < 0$) for all values of $r^w$.

(b) Radial acceleration $a_r$ of a particle (see Eq. (IV.39)) for $k_{\phi c} = k_{\phi c, \min}(r_{\max} = A/2)$ where $A = 2, 10^5$ pc has been chosen as typical for quasar-galaxy systems (see Table I in Section III).
\[ \Phi = \int \left[ \frac{1}{v_m^2} - \frac{1}{r^2} \frac{r}{r + A} \right] - \frac{2}{r} \frac{dr}{r(r + A)}. \] (IV.40)

When \( A \to 0 \) then the trajectory calculated according to above equation is the straight line \( r = r_m/(\cos \Phi) \) passing by the center at a distance of \( r_m \) (impact parameter). On the other hand light traveling in our spacetime (with \( A \neq 0 \)) is deflected even in absence of baryonic matter.

### D Redshift of the radiation from stable circular orbits.

Let us suppose for simplicity that distant observer is located far away from the center of the system at a distance far bigger than the size of the system, so its peculiar motion with respect to the center of the system is negligibly small. Let us also assume that the dynamical time scale \( t_{dyn} \) is greater than the characteristic timescale \( t_{obs} \) over which the observations are performed i.e. \( t_{dyn} > t_{obs} \). In such a case the observed motion of the “particle” is seen only as instantaneous redshift or blueshift. If the motion of a “particle” takes place along the stable circular orbit the Doppler shift is equal to (see Figure 4a)

\[ z_D = \sqrt{\frac{c + v_{st}^2 \sin \Phi(\tau)}{c - v_{st}^2 \sin \Phi(\tau)}} - 1, \] (IV.41)

where \( \Phi(\tau) = \int_0^\tau \Omega_{st}^2 d\tau \) (see Eq.(IV.18) and Eq.(IV.32)) and the angle \( \Phi(\tau) \) is counted from the direction to observer i.e. \( \Phi(\tau = 0) = 0 \) and \( v_{st}^2 \) is given by Eq.(IV.33).

Now the gravitational redshift according to Eq.(III.6) is equal to (see Figure 3b)

\[ z_g = \frac{\lambda_{obs}}{\lambda_s} - 1 = \frac{\omega_s}{\omega_{obs}} - 1 = \sqrt{\frac{r_s + A}{r_s}} - 1. \] (IV.42)

Figure 4: (a) Doppler shift (maximal) \( z_D \) caused by motion of a particle along a stable circular orbit (see Eq.(IV.41)) for different values of internal momentum. The curve \( z_g \) denotes the gravitational redshift (see Eq.(IV.42)). Analogously as in Figure 2 the continuous line has physical meaning only up to \( r_{w_{\text{max}}} = 1/2 \).

(b) The combined effect of gravitational and Doppler redshifts (see Eq.(IV.43)). Analogously as in Figure 2 the continuous line has physical meaning only up to \( r_{w_{\text{max}}} = 1/2 \).
It is not difficult to see that the combined effect of these redshifts is the following (see Figure 4b)

\[ z = (z_g + 1) (z_D + 1) - 1. \]  \hspace{2cm} (IV.43)

V. CONCLUSIONS AND PERSPECTIVES

In the present paper we have recognized certain non-perturbative six-dimensional "spherically" symmetric solutions of the Einstein equations. They are asymptotically flat but fundamentally different from the Schwarzschild solutions in four-dimensional spacetime. The motion of test particles has been analyzed. The solutions presented in Section III are parameterized by the parameter \( A \) which has similar dynamical consequences (especially for \( r \gg A \)) as the mass \( M = \frac{A c^2}{2G} \) — its existence would be perceived by an observer in the same way as invisible mass.

We may imagine that our six-dimensional world could be compactified in an non-homogeneous manner. In this picture the scalar "basic" field \( \varphi \) would form a kind of ground field\(^\text{13}\). The phenomenon of flat rotation curves and the connection of presented model to the so called dark matter problem will be a subject of separate paper.

However looking from the above mentioned perspective, we would like to invoke here two classes of phenomena. Some time ago Tanaka (1995) reported the detection of the relativistic effects in an X-ray emission line (the K\(\alpha \) line) from ionized iron in the galaxy MCG-6-30-15. The line is extremely broad, corresponding to a velocity of \( \sim 10^5 \text{ km/s} \approx 0.3 \, c \), and asymmetric, with most of the line flux being redshifted. This observation is not isolated since, in several objects, broad redshifted lines have been detected but no strong blue-shifted lines have been seen [23]. This is an argument against any asymmetrical-outflow hypothesis in which the flow is directed away from us because some objects should then have the flow directed towards us. On the other hand these observations have just been properly explained (Section IV) for the motion along circular orbits with the internal "total momentum" square \( k^{2}_\varphi < 0 \) because in the model even the whole Doppler effect connected with the blue wing moving towards us could be hidden bellow the gravitational redshift effect (see Figures 4a,b).

There are also other phenomena which call for explanation and are hard to understand from the point of view of the standard lore. One class of such problems is associated with the nature of the redshift of galaxies. There is a number of evidence that the redshifts are at least partially intrinsic properties of the galaxies, apparently quantized and time variable [24, 25, 26].

It has been known for a long time that there exist associations of quasars and galaxies where the components have widely discrepant redshifts [27]. We can think of a simple explanation of such apparently strange phenomena in terms of presented model which has nothing to do with standard cosmological interpretation of the red-

\(^\text{13}\) It is interesting to notice that recently there has been a strong tendency to draw analogies between cosmology and condensed matter physics [21, 22].
shift as a manifestation of the Hubble expansion. Let us imagine that a quasar and galaxy system is captured by the local configuration of the scalar field $\varphi$ – such like described in our model. It is natural to suppose that both the quasar and the galaxy are moving along the stable orbits and we may thus apply the results obtained in the Section IV. If the things are arranged so that the quasar is closer to the center of the $\varphi$ ground field configuration, it would have a (much) greater redshift than the galaxy located peripherally. As it has been illustrated on Figures 4a,b the whole idea works if the quasar is closer to the center than a fraction of $A$. At such distance the combined gravitational Doppler redshift given by Eq.(IV.43) is a steep function of $r$, which makes the idea working if the quasar–galaxy constitute a close binary system. Such a possibility has an attractive feature that bridges connecting galaxies and quasars may be explained as evidence of an infall of matter from the galaxy to the center of the field $\varphi$. Let us recall that the field $\varphi$ strongly accelerates the infalling particles (see Eq.(IV.39)), and having in mind our demand that the quasar is located closer to the center it may shed some light on the nature of quasar emission. The idea briefly outlined above deserves further deeper considerations. It would be also of interest to examine interactions of these dilatonic centers and their distribution in the universe.

The above mentioned observational facts [28] cannot find any reasonable explanation within the standard interpretation which claims that the Hubble expansion is responsible for the redshifts of galaxies. Hence, they require the search for models entirely different from evolutionary ones. The true view of the universe is not so faraway from us.

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APPENDIX A: THE SOLUTION FOR $A < 0$

If the parameter $A < 0$ then Eqs.(II.19)-(II.21) are valid only when $r > |A|$. The metric tensor becomes singular for $r = |A|$. However its determinant $g$ (see Eq.(II.26)) remains well defined. As in Section III we write the temporal and radial components of the metric $g_{MN}$ and the internal “radius” $\varrho(r)$ (see Eqs.(II.20) and (II.25))

$$
\begin{align*}
g_{tt} &= \frac{r}{r-|A|} \\
g_{rr} &= -\frac{r}{r-|A|} \\
\varrho(r) &= d_{\text{out}} \sqrt{\frac{r}{r-|A|}}.
\end{align*}
$$

(A1)

The formulae for the scalar curvature $\mathcal{R}$ (see Eq.(II.23)) and scalar field $\varphi$ Eq.(II.21) are

$$
\mathcal{R} = \frac{A^2}{2r^3(r-|A|)}
$$

(A2)

$$
\varphi(r) = \pm \sqrt{\frac{1}{2\kappa_0}} \ln\left(\frac{r}{r-|A|}\right),
$$

(A3)

where $d_{\text{out}}$ is the constant.
Now the physical radial distance $r_{l-A}$ from the the radius $|A|$ to the radius $r > |A|$ is equal to

$$r_{l-A} = \int_{|A|}^{r} dr \sqrt{-g_{rr}} = \sqrt{\frac{r}{r-|A|}} (r-|A|) + \left( A4 \right) + \frac{1}{2} |A| \ln \left( \frac{-|A|+2r+2(r-|A|) \sqrt{\frac{r}{r-|A|}}}{|A|} \right) > r-|A|. $$

Like in Section II we have $g_{tt} \to 1$ for $r \to \infty$ (see Eq.(III.1)). Hence comparing the gravitational potential $g_{tt} = \frac{r}{r-|A|} \approx 1 + \frac{|A|}{r}$ for $r \gg |A|$ with the gravitational potential $g_{tt} = 1 - \frac{G M}{r}$ for a field induced by the mass $M$ we notice, that gravitational potential $g_{tt}$ (see Eq.(A1)) is the repulsive one. This is the reason why this case has been omitted in the main text as till now unobserved but because it formally provides a solution to the model we reproduce some of its properties here.

Now the real radial distance $r_1$ from the center $r = 0$ to the point with $r > |A|$ is equal to (see Eq.(A4))

$$r_1 = |A| + r_{l-A} > r . \quad (A5)$$

Using Eq.(A1) we can rewrite Eq.(III.4) as follows

$$\frac{\omega_{obs}}{\omega_s} = \sqrt{\frac{r_{l-A}}{r_{obs}-|A|}} . \quad (A6)$$

As before, we take for simplicity the limit when the observer is in the infinity. So we get (see Figure 5)

$$\frac{\omega_{obs}}{\omega_s} = \sqrt{\frac{r_w}{r_s^w}} - 1 \quad \text{where} \quad r_s^w = \frac{r_s}{|A|} . \quad (A7)$$

We obtained the result that the nearer the source is to the surface given by the equation $r = |A|$ the more the emitted photon which reaches the observer is blueshifted.

\[\text{Figure 5: The ratio of the frequency } \omega_{obs} \text{ of the photon which reaches the observer to the frequency } \omega_s \text{ of the photon emitted from the source as a function of the relative distance } r_s^w = \frac{r_s}{|A|} \text{ of the source from the center of } \varphi \text{ field } (A < 0).\]

APPENDIX B: SCALE INVARIANCE

Let us rewrite the Hamilton-Jacobi equation (see Eq.(IV.2)) in the following way

$$\frac{r_w^w + 1}{r_w^w} \left( \frac{\partial S}{\partial t_w} \right)^2 - \frac{r_w^w + 1}{r_w} \left( \frac{\partial S}{\partial r_w} \right)^2 - \frac{1}{(r_w^w)^2} \left( \frac{\partial S}{\partial \phi} \right)^2 - \frac{1}{(d^w)^2} \left( \frac{\partial S}{\partial r} \right)^2 - \frac{1}{(d^w)^2} \frac{r_w^w}{r_w^w + 1} \cos^2 \theta \left( \frac{\partial S}{\partial \vartheta} \right)^2 - \frac{1}{(d^w)^2} \frac{r_w^w}{r_w^w + 1} \left( \frac{\partial S}{\partial \varpi} \right)^2 = 0 , \quad (B1)$$

where

$$r_w = \frac{r}{A}, \quad t_w = \frac{c t}{A}, \quad d^w = \frac{d}{A}, \quad l^w = m A . \quad (B2)$$

From the form of this equation we can notice that the model is explicitly scale invariant. It means that the systems with different values of $r, A, d, m$ and $t$ have the same physical properties provided the values of $l^w, r^w, t^w$ and $d^w$ are the same. In the other words this means that whenever the scalar field $\varphi$ is present the classical picture
of the world follows the same patterns from the micro to the marco-scale. The discussion of quantum aspects shall be presented in a separate paper but it is worth noting here that the Klein-Gordon equation for our six-dimensional model possesses similar scaling properties as the Hamilton-Jacobi equation above.

REFERENCES

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